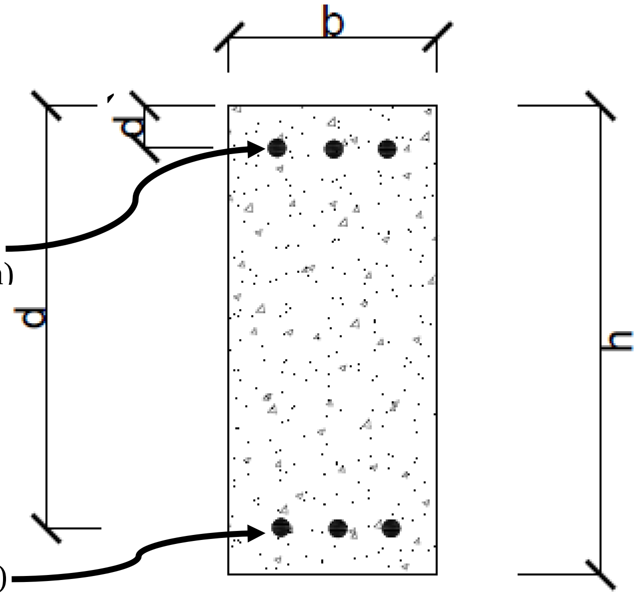
Analysis of rectangular beams with tension and compression reinforcements (Doubly reinforced beam)

The steel that is occasionally used on the compression sides of beams is called compression steel, and beams with both tensile and compressive steel are referred to as **doubly reinforced beams**.

When the beam cross section is limited because of architectural or other considerations. It may happen that the concrete cannot develop the compression force required to resist the given bending moment; in this case, reinforcement is added in the compression zone, resulting in doubly reinforcement beams.



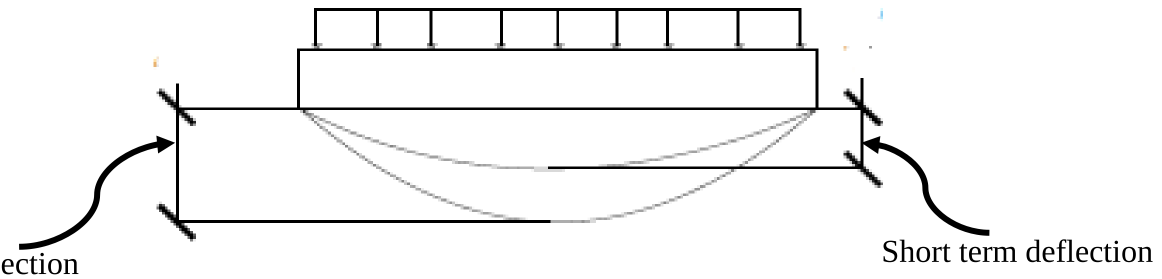
As` (Reinforcement area for compression)

As (Reinforcement area for tension)

- As is area for tension reinforcement, thus $\rho = \frac{As}{bd}$
- As` is area for compression reinforcement, thus $\rho` = \frac{As`}{bd}$

- There are four main reasons for using compression reinforcement in beams:

1. **Reduce sustained–load deflection (Long Term Deflection)**
   It has been found that the inclusion of some compression steel will reduce the long-term deflections of members.



Long term deflection

Short term deflection

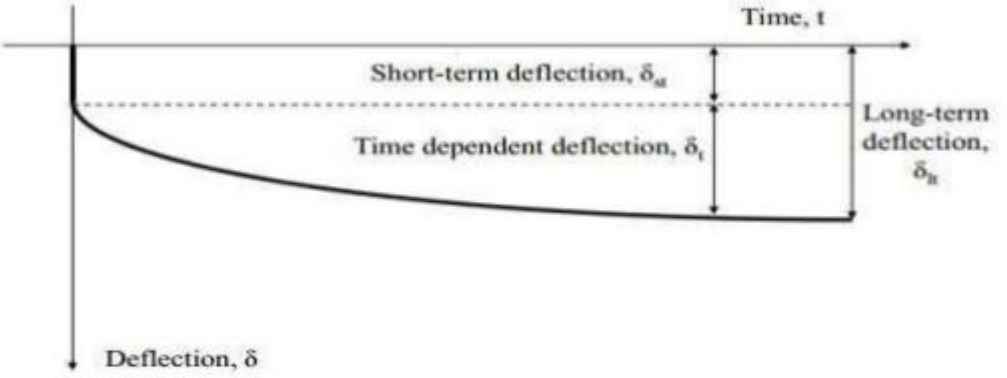
It is useful to note, there are two types of deflections:

a. Immediate (short) deflection

This deflection occurs immediately upon the application of a load.

b. Long term deflection

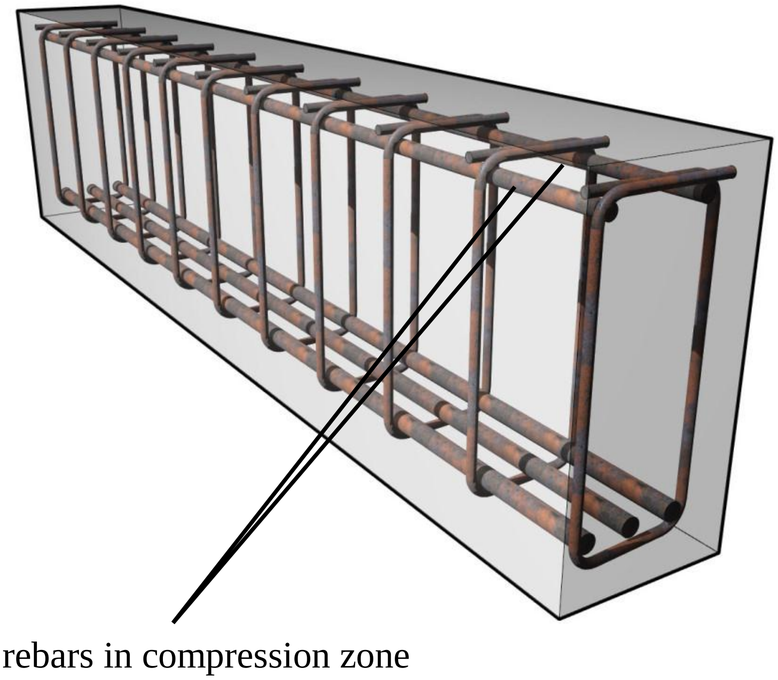
Take into account the shrinkage and creep movements.



- Calculation of deflection of beam and compare it with allowable limits in ACI Code is under serviceability requirements of beam and will be studying briefly in fourth year (senior course).

2. Stirrups Supports

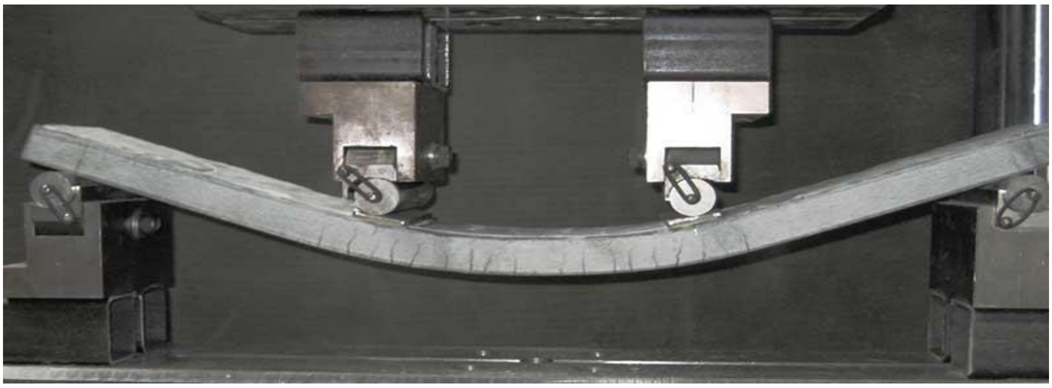
Continues compression bars are also helpful for positioning stirrups (by tying them to the compression bars) and keeping them in place during concrete placement and vibration.



Minimum rebars in compression zone
to support stirrups

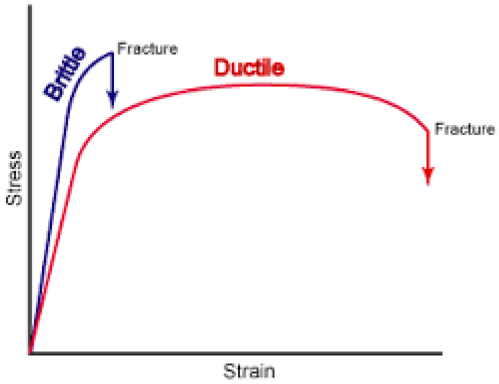
3. Increase ductility

Compression reinforcement increases not only the resisting moments of concrete sections but also the amount of curvature. This means that the ductility of such sections will be appreciably increased. Though expensive, compression steel makes beam tough and ductile, enabling them to withstand large moment's and stress reversals such as might occur during earthquakes.



Ductile concrete beam **(favorite)**

Brittle Concrete beam **(not favorite)**



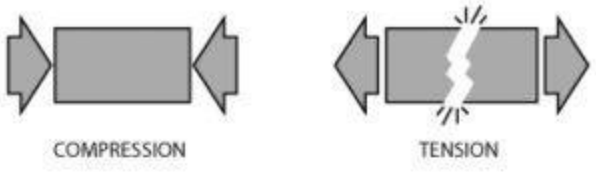
Stress-strain diagram for brittle and ductile materials.

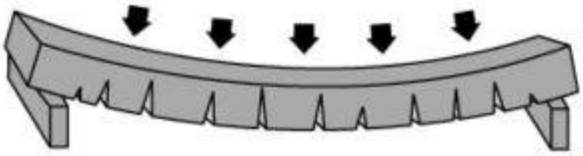
4. Changing the failure mode from compression to tension failure

According to ACI Code, all beams are to be designed for yielding of the tension steel, and thus $\rho \leq \rho_{max}$.

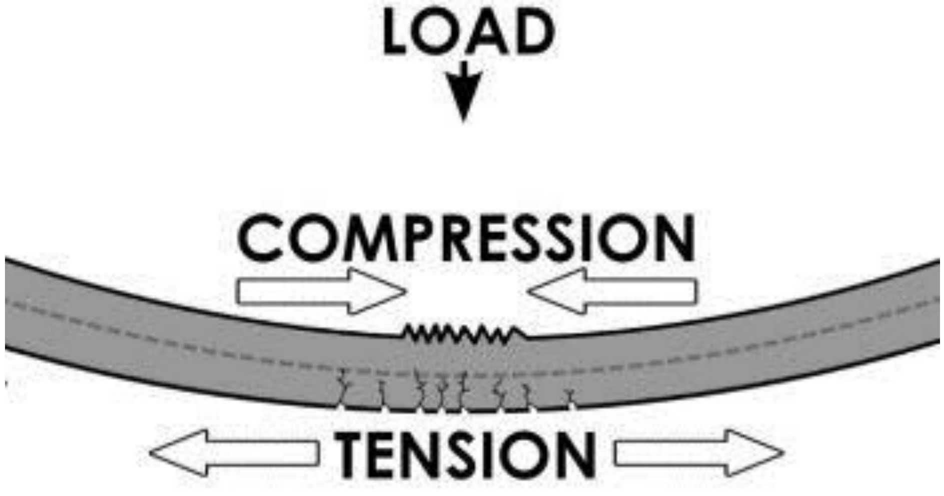
As was mentioned before sometimes beam cross section is limited because of architectural or other considerations. So addition steel in compression zone is required to be added to change the state of failure from compression to tension.

It is useful to remember there are two types of failure in concrete beam; the first one is the failure of tension zone of concrete and yielding of steel before compression zone and this type of failure is **required and permitted** in ACI code, consequently this mode of failure will give early notice for beam before failure which in turn give times for people to leave the building before collapse, however the second type of failure do not give any warning before failure and this happened when the compression zone fails before tension zone and this type is **not permitted** in ACI code.

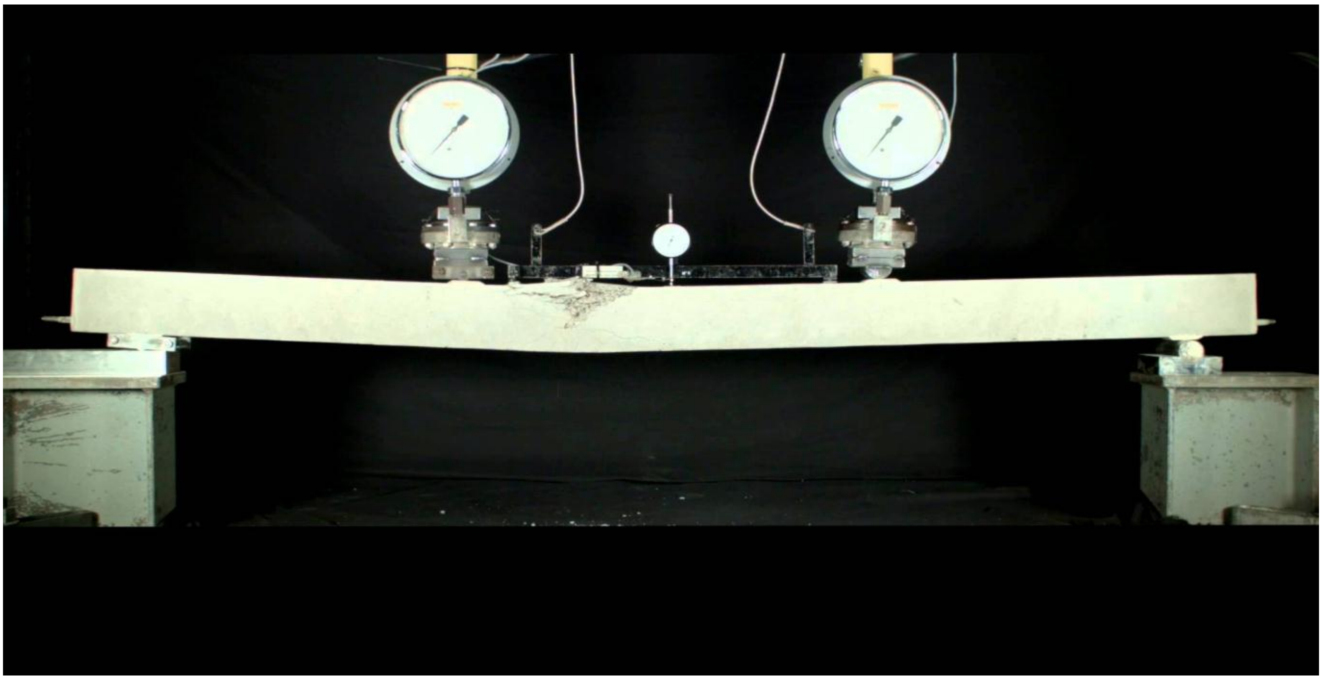




Tensile forces pull apart the bottom of this concrete slab when it bends



Tension failure of beam **(<u>permitted</u> in ACI Code)**



Compression failure of beam **(<u>not permitted</u> in ACI Code)**

Procedure Analysis for Rectangular Beams with tension and compression Reinforcements (A Doubly Reinforcement)

1. Check the reason for using of compression reinforcement

   find $\rho = \frac{As}{bd}$

   and $\rho_{max} = 0.85\beta_1 \frac{fc`}{fy} \frac{\epsilon u}{\epsilon u + 0.004}$ where $\epsilon_u = 0.003$

   If

   $\rho \le \rho_{max}$

   Then the compression reinforcement has been used either to reduce sustained-load deflection or to stirrups support or to increase ductility and it is effect can be neglected in the beam design.

   Then analysis the beam as singly reinforcement ■

   Else, if

   $\rho > \rho_{max}$

   Then the compression reinforcement has been used to change the mode of failure from compression to tension failure, and then this reinforcement must be included in the beam analysis then go to step 2.

2. Calculate $\rho`_{max}$

   $\rho`_{max} = \rho_{max} + \rho` \frac{f_s^`}{f_y}$

   $\rho` = \frac{As^`}{bd}$

   Where $f_s^`$ is stress in the **compression reinforcement**. It can be computed from relation below:

   $f_s^` = E_s \left[\epsilon_u - \frac{d^`}{d} (\epsilon_u + 0.004)\right] \le f_y$ where $E_s = 200{,}000$ Mpa and $\epsilon_u = 0.003$

   If $\rho \le \rho`_{max}$ O.k

   If $\rho > \rho`_{max}$ section is not O.k

3. Calculate $\rho_{cy}^`$

   $\rho_{cy}^` = 0.85\beta_1 \frac{fc`}{fy} * \frac{d`}{d} \frac{\epsilon u}{\epsilon u - \epsilon y} + \rho^`$

   If

   $\rho_{cy}^` \le \rho$ Then calculate the moment according to step 4.1

Else if

$\rho_{cy}^{`} > \rho$ go to step 4.2

4.1 Compute section nominal moment M_n when $(\rho_{cy}^{`} \le \rho)$

$$M_n=M_{n1}+M_{n2}=A_S^{`}f_y(d-d`) + (As-A_S^{`})f_y(d-\frac{a}{2})$$

$$a=\frac{(As-A_S^{`})*fy}{0.85fc`*b}$$

Calculate $\emptyset$

$$c=\frac{a}{\beta 1}$$

$\epsilon_t=\frac{dt-c}{c}\epsilon_u$ where: $\epsilon_u=0.003$

- If $\epsilon_t \ge 0.005$, then $\emptyset=0.9$
- If $\epsilon_t < 0.005$ then

  $\emptyset=0.483+83.3*\epsilon_t$

Calculate $\emptyset$Mn ■

4.2 Compute section nominal moment M_n when$(\rho_{cy}^{`} > \rho)$

$$M_n=M_{n1}+M_{n2}= 0.85fc`ab\,(d-\frac{a}{2}) +A_S^{`}f_s`\,(d-d`)$$

$fs` < fy$

Calculate fs`:

$$fs` = \epsilon_u *Es*(\frac{c-d`}{c})$$

Where c:

$$c = \sqrt{Q+R^2}-R$$

$$Q=\frac{600d`As`}{0.85\beta 1fc`b}$$

And

$$R=\frac{600As`-f_yAs}{1.7\beta_1fc`b}$$

$a=\beta_1 c$

Calculate $\emptyset$

$\epsilon_t=\frac{dt-c}{c}\epsilon_u$ where: $\epsilon_u=0.003$

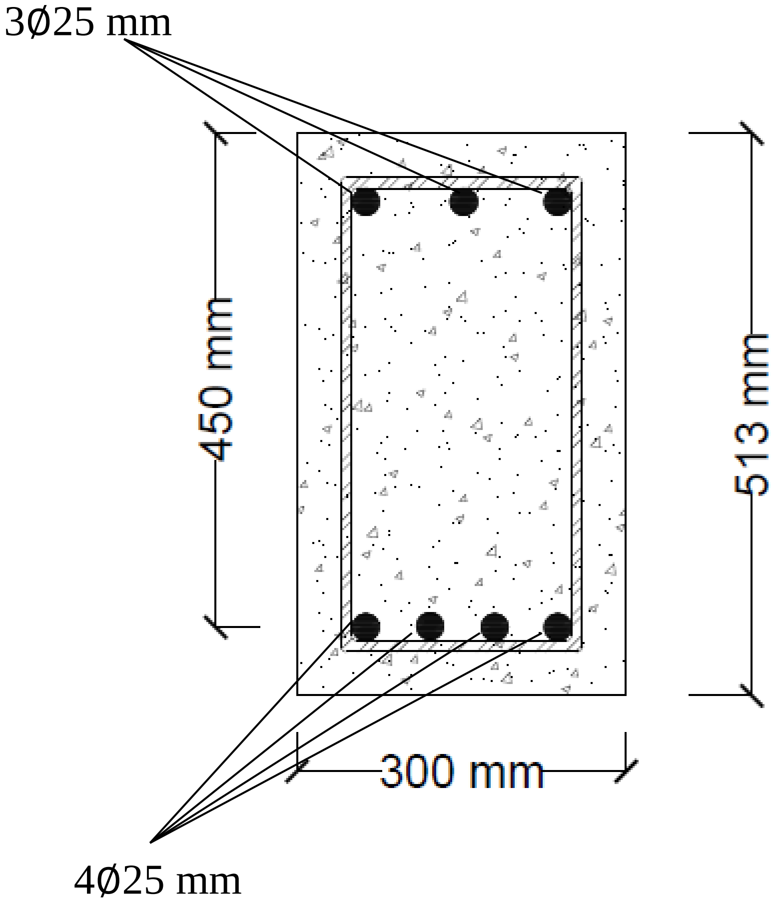
- If $\epsilon_t \ge 0.005$, then $\emptyset=0.9$
- If $\epsilon_t < 0.005$ then

  $\emptyset=0.483+83.3*\epsilon_t$

Calculate $\emptyset$Mn ■

Example1: Check the adequacy of beam shown in figure below and compute its design strength according to ACI Code. Use fc` = 20 MPa and fy= 300 Mpa

A $_{\text{bar of 25mm}}$ = 490mm^2



Solution:

1. Check the reason for using of compression reinforcement

find $\rho = \frac{As}{bd} = \frac{4*490}{300*450} = \frac{1960}{300*450} = 14.54 * 10^{-3}$

and $\rho_{max} = 0.85\beta_1 \frac{fc`}{fy} \frac{\epsilon u}{\epsilon u+0.004}$ where ϵu=0.003

$\rho_{max} = 0.85*0.85*\frac{20}{300}\frac{0.003}{0.003+0.004} = 20.6*10^{-3}$

$\rho < \rho_{max}$

Then the compression reinforcement has been used either to reduce sustained-load deflection or to stirrups support or to increase ductility and it is effect can be neglected in the beam design.

Then the beam can be analysis as singly reinforcement■

2. Calculate Ø

$$a=\frac{As*fy}{0.85fc`*b}=\frac{1960*300}{0.85*20*300}=115\text{ mm}$$

$$c=\frac{a}{\beta 1}=\frac{115}{0.85}=135\text{ mm}$$

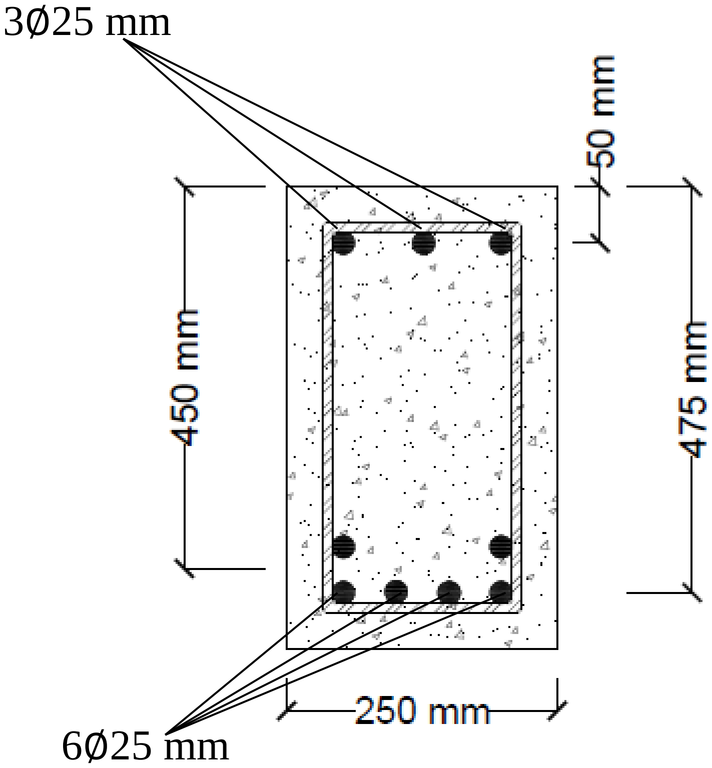
$$\epsilon_t=\frac{dt-c}{c}*\epsilon u \qquad \text{where: } \epsilon_u=0.003$$

$$\epsilon_t=\frac{450-135}{135}*0.003 \;=7*10^{-3}>0.005 \therefore\; Ø=0.9$$

3. Calculate $ØMn$

$$ØMn=ØAs*fy\,(d-\frac{a}{2})=0.9*1960*300*(450-\frac{115}{2})*10^{-6}=207\text{ kN.m}\ ■$$

Example 2 : Check the adequacy if beam shown in figure below and compute its design strength according to ACI Code. Use fc` = 20 MPa and fy= 300 Mpa Area $_{\text{bar of 25mm}}$ = 490mm^2



Solution:

1. Check the reason for using of compression reinforcement

find $\rho=\frac{As}{bd}=\frac{6*490}{250*450}=\frac{2940}{250*450}=26.1*10^{-3}$

and $\rho_{max}=0.85\beta_1\frac{fc`}{fy}\frac{\epsilon u}{\epsilon u+0.004}$ where $\epsilon_u=0.003$

$$\rho_{max}=0.85*0.85*\frac{20}{300}\frac{0.003}{0.003+0.004}=20.6*10^{-3}$$

$\rho > \rho_{max}$

Then the compression reinforcement has been used to change the mode of failure from compression to tension failure, and then this reinforcement must be included in the beam analysis ■

2. Calculate $\rho`_{max}$

$$\rho`_{max} = \rho_{max} + \rho` \frac{f_s`}{f_y}$$

$$\rho` = \frac{As`}{bd} = \frac{3*490}{250*450} = 13.1*10^{-3}$$

$$f_s` = E_s \left[\epsilon u - \frac{d`}{d}(\epsilon_u + 0.004)\right] \leq f_y \quad \text{where } E_s = 200{,}000 \text{ Mpa and } \epsilon_u = 0.003$$

$$f_s` = 200{,}000 \left[0.003 - \frac{50}{450}(0.003 + 0.004)\right] = 444 > fy$$

$\therefore$ fs`=fy=300 MPa

$$\rho`_{max} = \rho_{max} + \rho` \frac{f_s`}{f_y} = 20.6*10^{-3} + 13.1*10^{-3}\frac{300}{300} = 33.7*10^{-3}$$

$\rho \leq \rho`_{max}$ O.k

3. Calculate $\rho`_{cy}$

$$\rho`_{cy} = 0.85\beta_1 \frac{fc`}{fy} * \frac{d`}{d} \frac{\epsilon u}{\epsilon u - \epsilon y} + \rho`$$

$$\rho`_{cy} = 0.85*0.85*\frac{20}{300} * \frac{50}{450} \frac{0.003}{0.003 - \frac{300}{200{,}000}} + 13.1*10^{-3} = 23.8*10^{-3} < \rho$$

$\therefore$ fs`=fy=300 MPa

4. Compute section nominal moment M_n when ($\rho`_{cy} \leq \rho$)

$$M_n = M_{n1} + M_{n2} = A_S` f_y (d - d`) + (As - A_S`) f_y \left(d - \frac{a}{2}\right)$$

$$a = \frac{(As - A_S`)*fy}{0.85fc`*b} = \frac{(2940-1470)*300}{0.85*20*250} = 104 \text{ mm}$$

$$M_n = 1470*300(450-50) + (2940-1470)*300*\left(450 - \frac{104}{2}\right)$$

$$Mn = 176.4*10^6 \text{ kN.m} + 175.5*10^6 \text{ kN.m} = 352 \text{ kN.m}$$

Calculate $\emptyset$

$$c = \frac{a}{\beta 1} = \frac{104}{0.85} = 122 \text{ mm}$$

$$\epsilon t = \frac{dt - c}{c}\epsilon u = \frac{475-122}{122}*0.003 = 8.68*10^{-3} > 0.005 \therefore \emptyset = 0.9$$

Calculate $\emptyset$Mn = 0.9*352=317 kN.m ■

Example 3: recheck the adequacy of example 2 above but with d\` = 65 mm,

Solution :

1. Check the reason for using of compression reinforcement

   find $\rho = \frac{As}{bd} = \frac{6*490}{250*450} = \frac{2940}{250*450} = 26.1 * 10^{-3}$

   and $\rho_{max} = 0.85\beta_1 \frac{fc`}{fy} \frac{\epsilon u}{\epsilon u + 0.004}$ where $\epsilon_u = 0.003$

   $\rho_{max} = 0.85*0.85*\frac{20}{300}\frac{0.003}{0.003+0.004} = 20.6*10^{-3}$

   $\rho > \rho_{max}$

   Then the compression reinforcement has been used to change the mode of failure from compression to tension failure, and then this reinforcement must be included in the beam analysis ■

2. Calculate $\rho`_{max}$

   $\rho`_{max} = \rho_{max} + \rho` \frac{f_s`}{f_y}$

   $\rho` = \frac{As`}{bd} = \frac{3*490}{250*450} = 13.1 * 10^{-3}$

   $f_s` = E_s \left[\epsilon_u - \frac{d`}{d}(\epsilon_u + 0.004)\right] \leq f_y$ where E_s=200,000 Mpa and ϵ_u=0.003

   $f_s` = 200{,}000 \left[0.003 - \frac{65}{450}(0.003 + 0.004)\right] = 398 > fy$

   $\therefore fs` = fy = 300$ MPa

   $\rho`_{max} = \rho_{max} + \rho` \frac{f_s`}{f_y} = 20.6*10^{-3} + 13.1*10^{-3} * \frac{300}{300} = 33.7*10^{-3}$

   $\rho \leq \rho`_{max}$ O.k

3. Calculate $\rho`_{cy}$

   $\rho`_{cy} = 0.85\beta_1 \frac{fc`}{fy} * \frac{d`}{d} \frac{\epsilon u}{\epsilon u - \epsilon y} + \rho`$

   $\rho`_{cy} = 0.85*0.85*\frac{20}{300} * \frac{65}{450} \frac{0.003}{0.003 - \frac{300}{200{,}000}} + 13.1*10^{-3}$

   $\rho`_{cy} = 13.9*10^{-3} + 13.1*10^{-3} = 27*10^{-3} > \rho$

   $\therefore fs` < fy$

4. Compute section nominal moment M_n when($\rho`_{cy} > \rho$)

   $M_n = M_{n1} + M_{n2} = 0.85fc`ab\,(d - \frac{a}{2}) + A_S` f_s`\,(d - d`)$

   Calculate fs\`:

$fs` = \epsilon_u * Es * (\frac{c-d`}{c})$

Where c:

$c = \sqrt{Q + R^2} - R$

$Q = \frac{600d`As`}{0.85\beta 1fc`b} = \frac{600*65*1470}{0.85*0.85*20*250} = 15870$

And

$R = \frac{600As` - fyAs}{1.7\beta 1fc`b} = = \frac{600*1470-300*2940}{1.7*0.85*20*250} = 0$

$c = \sqrt{Q + R^2} - R = \sqrt{15870 + 0^2} - 0 = 126$ mm

$fs` = 0.003 * 200{,}000 * (\frac{126-65}{126}) = 290$ MPa $<$ fy O.k

$a = \beta_1 c = 0.85*126 = 107$ mm

$M_n = M_{n1} + M_{n2} = 0.85fc`ab\ (d - \frac{a}{2}) + \grave{A_S} f_s`\ (d-d`)$

$M_n = 0.85*20*107*250\ (450 - \frac{107}{2}) + 1470 * 290 * (450-65)$

$M_n = 180.3 * 10^6$ N.m + $164.1 * 10^6$ N.mm $= 344 * 10^6$ N.mm $= 344$ kN.m

Calculate ∅

$\epsilon_t = \frac{dt-c}{c}\epsilon_u$ where: $\epsilon_u = 0.003$

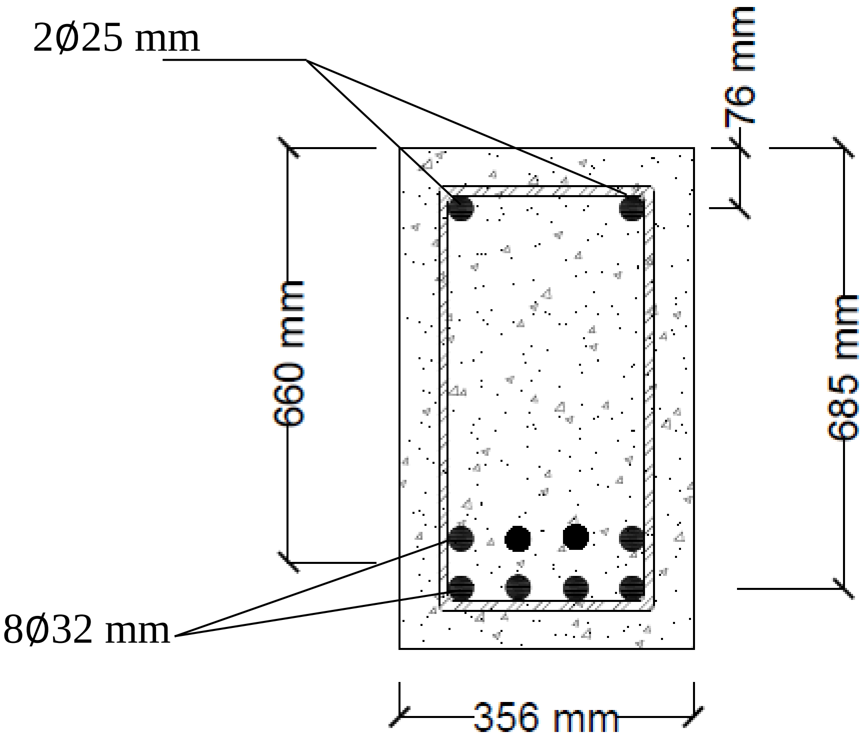
$\epsilon_t = \frac{475-126}{126} * 0.003 = 8.3*10^{-3} > 0.005\ \therefore\ \emptyset = 0.9$

Calculate ∅ Mn

∅ Mn=0.9*344=310 kN.m ■

Example 4: Check the adequacy of the beam shown below and compute its design strength according to ACI Code. Assume that:

1. fc` = 34.5 MPa
2. fy=414 MPa
3. Area of bar No.25mm=510 mm^2
4. Area of bar No.32mm=819 mm^2



Solution:

1. Check the reason for using of compression reinforcement

   find $\rho = \frac{As}{bd} = \frac{8*819}{356*660} = \frac{6552}{356*660} = 27.9 * 10^{-3}$

   and $\rho_{max} = 0.85\beta_1 \frac{fc`}{fy} \frac{\epsilon u}{\epsilon u + 0.004}$ where $\epsilon_u = 0.003$

   $\rho_{max} = 0.85*0.804*\frac{34.5}{414} \frac{0.003}{0.003+0.004} = 24.4*10^{-3}$

   $\rho > \rho_{max}$

   Then the compression reinforcement has been used to change the mode of failure from compression to tension failure, and then this reinforcement must be included in the beam analysis ■

2. Calculate $\rho`_{max}$

   $\rho`_{max} = \rho_{max} + \rho` \frac{f`_s}{f_y}$

$\rho' = \frac{As'}{bd} = \frac{2*510}{356*660} = \frac{1020}{356*660} = 4.34*10^{-3}$

$f_s' = E_s \left[\epsilon_u - \frac{d'}{d}(\epsilon_u + 0.004)\right] \le f_y$ where E_s=200,000 Mpa and ϵ_u=0.003

$f_s' = 200{,}000 \left[0.003 - \frac{76}{660}(0.003 + 0.004)\right] = 438.78 \text{ MPa} > 414 \text{ MPa}$

$\therefore fs' = fy = 414$ MPa

$\rho'_{max} = \rho_{max} + \rho' \frac{f_s'}{f_y}$

$\rho'_{max} = 24.4*10^{-3} + 4.34*10^{-3} = 28.9*10^{-3}$

$\rho < \rho'_{max}$ O.k

3. Calculate ρ'_{cy}

$\rho'_{cy} = 0.85\beta_1 \frac{fc'}{fy} * \frac{d'}{d} \frac{\epsilon u}{\epsilon u - \epsilon y} + \rho'$

$\rho'_{cy} = 0.85*0.804*\frac{34.5}{414} * \frac{76}{660} \frac{0.003}{0.003 - \frac{414}{200{,}000}} + 4.34*10^{-3} = 25.5*10^{-3}$

$\rho'_{cy} < \rho$

$\therefore fs' = fy = 414$ MPa

4. Compute section nominal moment M_n when ($\rho'_{cy} \le \rho$)

$M_n = M_{n1} + M_{n2} = A_S' f_y (d - d') + (As - A_S') fy (d - \frac{a}{2})$

$a = \frac{(As - A_S')*fy}{0.85fc'*b} = \frac{(6552-1020)*414}{0.85*34.5*356} = 219$ mm

$M_n = 1020*414*(660-76) + (6552-1020)*414*(660 - \frac{219}{2})$

$Mn = 247*10^6$ N.mm $+ 1261*10^6$ N.mm $= 1508$ kN.m

Calculate Ø

$c = \frac{a}{\beta 1} = \frac{219}{0.804} = 272$ mm

$\epsilon_t = \frac{dt - c}{c}\epsilon_u$ where: $\epsilon_u = 0.003$

$\epsilon_t = \frac{685-272}{272} * 0.003 = 4.55*10^{-3} < 0.005$

Ø=0.483+83.3*ϵ_t = 0.86

Calculate ØMn

ØMn = 0.86*1508 =1297 kN.m ■